Spatial-temporal correlations in the process to self-organized criticality

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A different type of spatial-temporal correlation in the process approaching the self-organized criticality is investigated for the two simple models for biological evolution. The changed behaviors of the position with minimum barrier are shown to be quantitatively different in the two models. Different results of the correlation are given for the two models. We argue that the correlation can be used, together with the power-law distributions, as criteria for self-organized criticality.

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The phenomenon of ‘‘self-organized criticality’’ (SOC), with potential applications ranging from the behavior of sandpile and the description of the growth of surfaces to generic description of biological evolution, has become as a topic of considerable interest [1–8]. It is observed that the dynamics of complex systems in nature does not follow a smooth, gradual path, instead it often occurs in terms of punctuations, or ‘‘avalanches’’ in other word. The appearance of the spatial-temporal complexity in nature, containing information over a wide range of length and time scale, presents a fascinating but longstanding puzzle. Such complexity also shows up in simple mathematical models for biological evolution and growth phenomena far from equilibrium. In former studies, power-law distributions for the spatial size and lifetime of the ‘‘avalanches’’ have been observed in various complex systems and are regarded as ‘‘fingerprints’’ for SOC. It seems that there is no general agreement on a suitable definition of SOC [9,10], although a minimal definition was given in [11]. Because there is no universally accepted ‘‘black-box’’ tests for the presence or absence of SOC based solely on observables, systems with a wide range of characteristics have all been designated as ‘‘self-organized critical.’’

While numerous numerical studies have claimed SOC to occur in specific models, and although the transition to the SOC state was studied in [12–14], a question has never been answered: How is the process approaching to the final dynamical SOC attractor characterized? One may even ask whether the phenomenon SOC can be adequately characterized by such power-law distributions. The answer to the latter question seems to be negative, as concluded in [15]. In Ref. [15] were pointed out ‘‘some striking observable differences between two ‘self-organized critical’ models which have a remarkable structural similarity.’’ The two models, as called the Bak-Sneppen (BS) models, are introduced in [16–18] and are used to mimic biological evolution. The models involve a one-dimensional random array on L sites. Each site represents a species in the ‘‘food chain.’’ The random number (or barrier) assigned to each site is a measure of the ‘‘survivability’’ of the species. Initially, the random number for each species is drawn uniformly from the interval (0, 1). In each update, the least survivable species (the update center) and some others undergo mutations and obtain new random numbers which are also drawn uniformly from (0, 1). In the first version of the model (the local or nearest-neighbor model), only the update center and its two nearest neighbors participate the mutations. In the second version, $K-1$ other sites chosen randomly besides the update center are involved in the update and assigned new random survivabilities (so this version is called random neighbor model). Periodic boundary conditions are adopted in the first model. As shown in [18–20], the second version is analytically solvable. Investigation in [15] shows that some behaviors of the local and random neighbor models are qualitatively identical. They both have a nontrivial distribution of barrier heights of minimum barriers, and each has a power-law avalanche distribution. But the spatial and temporal correlations between the minimum barriers show different behaviors in the two models and thus can be used to distinguish them.

In all the studies mentioned above, spatial and/or temporal distributions of the ‘‘avalanches’’ and correlations between positions with minimum of barriers are investigated separately. As shown in many studies, however, spatial and/or temporal distribution of the ‘‘avalanches’’ alone cannot be used as a criterion for SOC, nor can the spatial or temporal correlation do. In this Brief Report, we attempt to study a different kind of correlation between minimum barriers in the process of the updating in the two models for biological evolution. The correlation between the positions with minimum barriers at time (or update) s and $s+1$ is investigated. Since the correlation involves *two sites* at *different* times, it is of spatial-temporal type. Thus it may be suitable for the study of spatial-temporal complexity.

Consider the update process of the local neighbor model. Initially, each site is assigned a random number. All the random numbers are drawn uniformly from interval (0,1). Denote $X(s)$ the site number with minimum barrier after s updates. The sites can be numbered such that $1 \leqslant X(s) \leqslant L$. To see how $X(s)$ changes in updating process in the model $X(s)$ is shown in Fig. 1 as a function of s for an arbitrary update process for lattice size $L=200$ with s from 1 to 2000. The lower part of Fig. 1 is a zoomed part of the upper one for small s. It is clear that $X(s)$ seems to be random when s is small. With the going-on of updating, $X(s)$ becomes more and more likely to be in the neighborhood of last update center, $X(s-1)$. So there appear some plateaulike parts in Fig. 1. In other words, there appears to be some correlation between $X(s)$ when the system is self-organized to approach the criti-

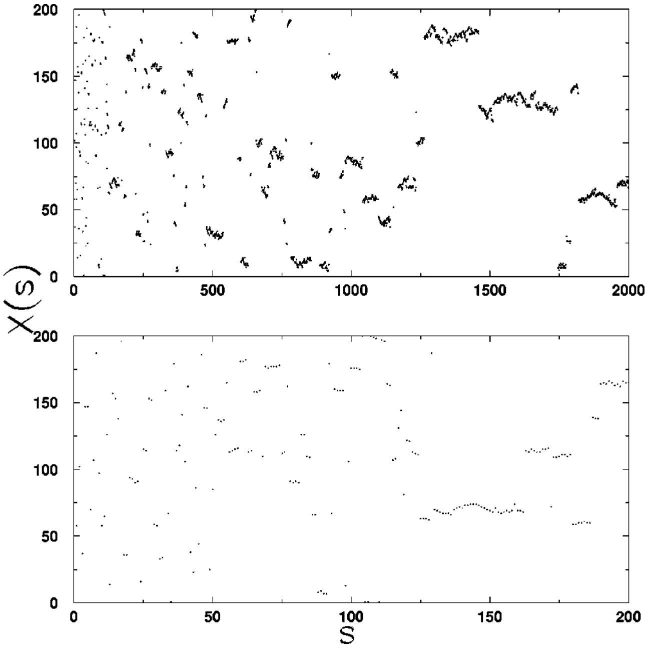

FIG. 1. The change of site $X(s)$ with time s for an arbitrary event in the nearest neighbor version of the BS model for biological evolution.

cal state. So, it may be fruitful to study the self-correlation of $X(s)$ in searching quantities characterizing the process to SOC. For this purpose, one can define a quantity

$$C(s)=\langle X(s)X(s+1)\rangle-\langle X(s)\rangle\langle X(s+1)\rangle, \tag{1}$$

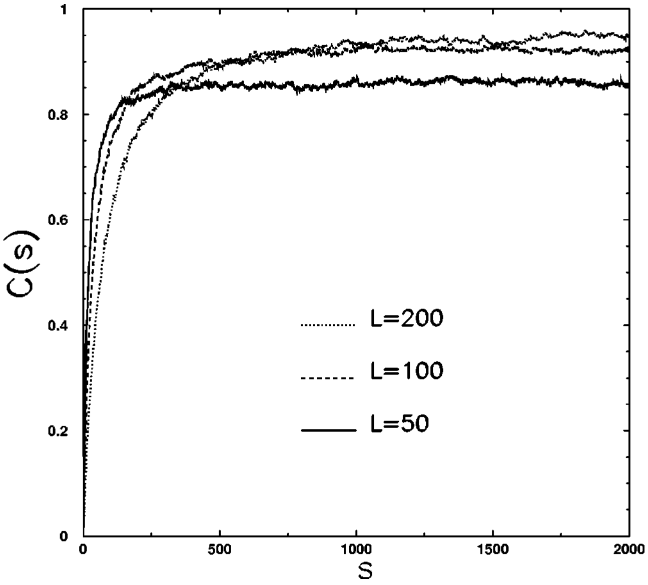

FIG. 2. The correlation function $C(s)$ as a function of s for lattice size $L=50$, 100, and 200 for the same model as in Fig. 1.

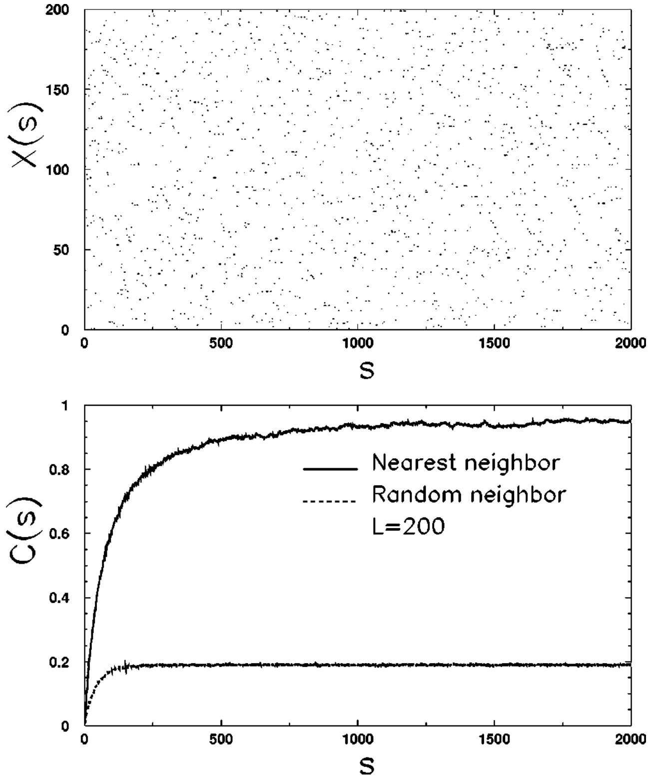

FIG. 3. Upper part: The change of site $X(s)$ with s for the random neighbor version of the BS model for biological evolution. Lower part: The correlation function $C(s)$ for the two versions as functions of s for $L=200$.

with average over different events of updating. Obviously, if there is no correlation between the sites with minimum barrier at time s and $s+1$, or $\langle X(s)X(s+1)\rangle=\langle X(s)\rangle\langle X(s+1)\rangle$, $C(s)$ will be zero. Thus, $C(s)$ can show whether there is correlation between $X(s)$ and also give a measure of the strength of the correlation. Because of the randomness of the survivability at each site, $X(s)$ can be $1,2,\ldots,L$ with equal probability, $1/L$. Thus, $\langle X(s)\rangle=(L+1)/2$ for every time s. It should be pointed out that $\langle X(s)\rangle=(L+1)/2$ does not mean any privilege of sites with numbering about (L +1)/2. In fact, all sites can be the update center with equal chance at time s if the update process is repeated many times from the initial state. Due to the randomness of the updated survivability $X(s+1)$ can also take any integer from 1 to L. However, the distribution of $X(s+1)$ is peaked at $X(s)$ when s is large, see [13] for detail. With the update going on, the width of the distribution becomes more and more narrower. When the width becomes narrow enough, $\langle X(s)X(s+1)\rangle$ will turn out to be $\langle X^2(s)\rangle=(2L^2+3L+1)/6$. So, $C(s)$ will approach $(L^2-1)/12$ for large s. In above definition for $C(s)$, however, the neighboring relation between $X(s)$ and $X(s+1)$ cannot be realized once the numbering for the sites is given. Due to the periodic boundary conditions adopted in the model, one of the nearest neighbors of the site with numbering 1 is the one numbered L. To overcome this shortcoming, one can introduce an *orientational shorter distance* $\Delta(s)$ between $X(s)$ and $X(s+1)$. Imagine the L sites with numbering $1,2,\ldots,L$ are placed on a circle in clockwise order. Then $|\Delta(s)|$ is the shorter distance between the two sites on the circle. If $X(s+1)$ is reached along the shorter curve from $X(s)$ in clockwise direction, $\Delta(s)$ is positive. Otherwise $\Delta(s)$ is negative. For definiteness, one can assume $-L/2\leqslant\Delta(s)<L/2$. With $\Delta(s)$, one can use

$$X'(s+1)=X(s)+\Delta(s) \tag{2}$$

in place of $X(s+1)$ in the definition of $C(s)$. Since $X'(s)$ can cross the (nonexisting) boundary between 1 and L and reflect the neighboring relation with $X(s)$, the effect of periodic boundary conditions on the correlation can be taken into account. [In the simulation of the BS model numbering the L sites with integer numbers $1,2,\ldots,L$ is necessary, but the start position can be arbitrary. Different numbering scheme will give the same results for $C(s)$, as physically demanded. This in return is also an indication of the equivalence of all sites in the presence of periodic boundary conditions.] To normalize the dependence of $C(s)$ on the size of the one-dimensional array, we can renormalize $C(s)$ by $(L^2-1)/12$. In the following, we use a normalized definition of $C(s)$:

$$C(s)=\frac{\langle X(s)X'(s+1)\rangle-\langle X(s)\rangle\langle X'(s+1)\rangle}{(L^2-1)/12}. \qquad (3)$$

In current study $X(s)$ and $\Delta(s)$ are determined from Monte Carlo simulations, and 500 000 simulation events are used to determine the averages involved. For each event, 2000 updates are performed from an initial state with random barriers on the sites uniformly distributed in (0, 1). The normalized correlation function $C(s)$ is shown as a function of s in Fig. 2 for $L=50$, 100, and 200. One can see that $C(s)$ is a monotonously increasing function of time s. As in our naive consideration, $C(s)$ is very small in the early stage of updates and becomes larger and larger for larger s, indicating the increase of the strength of correlation between the sites with minimum barrier at different times. The behavior of $C(s)$ with s exhibits different characteristics for small and large s. $C(s)$ increases with s very quickly for small s, but the rate becomes quite slow after a knee point. The knee point appears earlier for smaller L, showing the existence of a finite-size effect. Also, the seemly saturating value of $C(s)$ depends on the size L of the lattice, or more clearly, it increases with the lattice size L. Since only 500,000 simulation events are used in current study, there shows the effect of fluctuations in the figure.

The correlation between $X(s)$ can be investigated for the random neighbor model for biological evolution in the same way. For simplicity only the case with $K=3$ is taken into account. The generalization to other cases is straightforward. First, one can have a look on how $X(s)$ changes with update. $X(s)$ is shown as a function of s in the upper part of Fig. 3. This plot may look as a random scatter of points at first sight. But it is not. A close look reveals correlations: $X(s)$ often has almost same value for several consecutive or almost consecutive s values. However, no obvious plateaulike part can be seen in the figure, showing the difference between the two versions of the BS model. $C(s)$ is also studied and shown in the lower part of Fig. 3 as a function of s for the lattice size $L=200$. In the random neighbor version of the BS model, sites numbered with 1 and L are no longer neighbors. So, in the calculation of $C(s)$ from Eq. (3), $X(s+1)$ is used instead of $X'(s+1)$. The counterpart for the nearest neighbor model is also drawn in the figure for comparison. One can see that the saturating value is much smaller than in the case of the local neighbor version of the model.

From the discussions above one can see that the correlation between the sites with minimum barrier may play an important role in investigating SOC. The power-law distributions for the size and lifetime of the ''avalanches'' together with the different kind of correlation may be used as criteria for SOC.

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